Worksheet 7: November 13th

Problem 1. This problem walks through finding the Jordan normal form of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

A.) Compute the eigenvalues of the matrix either through a linear system or using $\det(A - xI) = 0$ (you should have two eigenvalues).

Solution: The determinant is $(1-x)^2(3-x)^2$, giving eigenvalues 1 and 3.

B.) Compute the dimension of the eigenspaces corresponding to each eigenvalue, i.e. $\dim(\ker(A - \lambda I))$.

Solution: Notice that

$$A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

each have 3 pivots, so both null spaces have dimension 1.

C.) Compute $\dim(\ker((A - \lambda I)^2))$ for each eigenvalue. Do we need to compute any higher powers? Why or why not?

Solution: Since we know that the generalized eigenspaces are each 2-dimensional, we must have 1.) that each of these null spaces have dimension 2 (otherwise, the chain of null spaces would be stationary and we could only have a 1D generalized eigenspace) and 2.) that we cannot reach a higher dimension.

D.) Use these values to compute the size of the Jordan blocks and the Jordan normal form.

Solution: From our formulas in both cases, we have 1 Jordan block of size 2 for each eigenvalue, giving JNF

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

E.) Lastly, compute the Jordan basis

Solution: First, we find v_1 so $(A - I)v_1 = 0$, then find v_2 so $(A - I)v_2 = v_1$. We do the same with $(A - 3I)$ to obtain v_3, v_4. This gives

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

Problem 2. Give an example of an operator whose characteristic polynomial is $(z-1)^2(z-2)^3$ and whose minimal polynomial is $(z-1)^2(z-2)$.
Solution: Any operator with 3 blocks $J_1(2)$ and 1 block $J_2(1)$ works.

Problem 3. Let V be a finite-dimensional complex vector space and $T \in L(V)$ have $\dim(\ker(T)) = 3$ and $\dim(Im(T)) = 4$. If $T^2 = T$, what is the characteristic polynomial of T?
Solution: Since $m_T(x)|x(x-1)$, T is diagonalizable with eigenvalues 0 and 1. Further, since $\ker(T^2_{=} \ker(T)$, $G(0,T) = \ker(T)$, so we have algebraic multiplicity 3 for 0. Lastly, by Rank-Nullity, V has dimension 7, so

$$p_T(x) = x^3(x-1)^4$$

B.) Show further that $tr(T) = 4$.
Solution: This is the sum of the eigenvalues.

Problem 4. Let T be a linear operator on a finite-dimensional complex vector space with minimal polynomial

$$m_T(x) = (x-\lambda_1)^{a_1}...(x-\lambda_k)^{a_k}$$

Prove that a_i is the size of the largest Jordan block associated to λ_i.
Partial Solution: Show that $(J_k(\lambda) - \lambda I)^k = 0$ but $(J_k(\lambda) - \lambda I)^{k-1} \neq 0$.